ALGEBRAIC GEOMETRY–FINAL EXAM
B. MATH. III
TIME : 3 HOURS, MARKS : 80

All rings are assumed to be commutative with identity. All varieties are defined over an algebraically closed field k, unless specified otherwise.
You are allowed to bring two pages of hand written notes.

(1) (a) Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then either $M = 0$ or $N = 0$.
(b) Let A be a ring and M and N finitely generated A-modules. Show that $\text{Supp}(M \otimes_A N) = \text{Supp}(M) \cap \text{Supp}(N)$. [7+3]

(2) Let A be a ring.
(a) Let $\{M_i\}_{i\in I}$ be a family of A-modules and let $M = \oplus_{i\in I} M_i$. Show that M is flat iff each M_i is flat.
(b) $A[x]$ is flat as an A-algebra. [7+3]

(3) Prove that $A[[X]]$ is a Noetherian ring iff A is a Noetherian ring. [10]

(4) Let $A \subseteq B$ be rings, B integral over A.
(a) If $a \in A$ is a unit in B then it is a unit in A.
(b) The polynomial ring $B[x]$ is integral over $A[x]$. [3+7]

(5) Let A be a ring such that each non-zero element of A is contained in only finitely many maximal ideals of A and for every maximal ideal $\mathfrak{m}$, the ring $A_\mathfrak{m}$ is Noetherian. Show that A is Noetherian. [10]

(6) Let A be a Noetherian local ring. Show that $\dim A[[X]] = \dim A + 1$.
Prove the same when A is any Noetherian ring, not necessarily local. [6+4]

(7) Let $\phi : X \to Y$ be a morphism between varieties.
(a) For each $P \in X$, ϕ induces a homomorphism of local rings $\phi_P^* : \mathcal{O}_{\phi(P),Y} \to \mathcal{O}_{P,X}$.
(b) ϕ is an isomorphism iff ϕ is a homeomorphism and the induced map ϕ_P^* on local rings is an isomorphism, for all $P \in X$. [3+7]

(8) Let P be a point on a variety X. Then $\dim \mathcal{O}_{P,X} = \dim X$. [10]

(9) (a) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

(b) Let $Y \subseteq \mathbb{A}^3$ be the algebraic set defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe the corresponding prime ideals. [3+7]

(10) Let H_i and H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbb{P}^n \setminus (H_i \cap H_j)$ is constant.
Using this or otherwise, show that any regular function on $\mathbb{P}^n$ is constant. [7+3]

STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE,, BANGALORE, INDIA-560059
E-mail address: souradeep_vs@isibang.ac.in